Worksheet 2 Numerical Analysis Fall 2024

Name: ______________________ NetID: | | | | | | | | | | | |

Name: ______________________ NetID: | | | | | | | | | | | |

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Work in groups of 2. Move your chairs if needed.

$$\mathbf{A} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix} \begin{bmatrix} - & \mathbf{v}_1^\top & - \\ - & \mathbf{v}_2^\top & - \\ & \vdots & \\ - & \mathbf{v}_n^\top & - \end{bmatrix}, \qquad \mathbf{B} = \sum_{i=1}^{n} \sigma_i \mathbf{u}_i \mathbf{v}_i^\top.$$

Problem 1. What is $\mathbf{A}\mathbf{v}_i$? What about $\mathbf{B}\mathbf{v}_i$? Does this mean $\mathbf{A} = \mathbf{B}$?

Problem 2.

1. Suppose $\mathbf{X}$ is a $n \times n$ matrix. Write $\|\mathbf{X}\|_\mathsf{F}^2$ in terms of the column-norms $\|[\mathbf{X}]_{:,i}\|_2^2$.

2. Suppose $\mathbf{X}$ is a $n \times n$ matrix and $\mathbf{U}$ is a $n \times n$ orthogonal matrix ($\mathbf{U}^\mathsf{T}\mathbf{U} = \mathbf{I}$). Show that $\|\mathbf{U}\mathbf{X}\|_\mathsf{F} = \|\mathbf{X}\|_\mathsf{F}$. Hint: use that $\|\mathbf{U}\mathbf{x}\|_2 = \|\mathbf{x}\|_2$ for any vector $\mathbf{x}$.

3. Let $\mathbf{A}$ be a $n \times n$ matrix with SVD $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\mathsf{T}$. Prove that $\|\mathbf{A}\|_\mathsf{F} = \sqrt{\sigma_1^2 + \cdots + \sigma_n^2}$, where σ_i are the singular values of $\mathbf{A}$. Hint: what is $\|\mathbf{\Sigma}\|_\mathsf{F}$?