

## Homework 6

## Numerical Analysis Fall 2024

### Instructions:

- Due 11/25 at 6:00pm on Gradescope.
- You must follow the submission policy in the syllabus

**Problem 1.** Suppose we have time-series data  $(t_1, y_1), \dots, (t_n, y_n)$ . We can try to fit the data with a polynomial of degree  $k$ . I.e. find a polynomial

$$p(x) = c_0 + c_1x + \dots + c_kx^k$$

so that at each time  $t_i$ , we have

$$p(t_i) \approx y_i.$$

To do this, we can solve a least squares problem

$$\min_{c_0, \dots, c_k} \sum_{i=1}^n (y_i - p(t_i))^2 = \min_{c_0, \dots, c_k} \sum_{i=1}^n (y_i - (c_0 + c_1t_i + \dots + c_kt_i^k))^2.$$

As we saw in class, this can be written as a linear algebra problem:

$$\min_{\mathbf{c} \in \mathbb{R}^{k+1}} \|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2^2$$

where

$$\mathbf{b} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 1 & t_1 & t_1^2 & \dots & t_1^k \\ 1 & t_2 & t_2^2 & \dots & t_2^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_n & t_n^2 & \dots & t_n^k \end{bmatrix}.$$

Get the data `temp.npy` from the website. Load the data:

```
temp = np.load('gdrive/MyDrive/na_f2024/hw files/temp.npy')
t = np.arange(190)/24 # time in days
```

We can plot the data:

```
plt.subplots(1,1,figsize=(12,4))
plt.plot(t,temp,marker='.',ls='None',label='data')
plt.ylabel('temperature (F)')
plt.xlabel('time since Oct 29 (days)')
plt.legend()
```

- (a) For each  $k = 5, 10, 15, 20, 25$ , set up the least squares problem for and solve it (either using `np.linalg.lstsq` or using a QR factorization followed by a triangular solve).

Add each of the polynomials (evaluated at a finer grid of  $t$  values) to the plot. Make sure they are labeled.

- (b) Note that we could represent our polynomial in terms of a different basis. I.e. instead of  $1, x, x^2, \dots, x^k$ , we could use any family  $p_0(x), p_1(x), \dots, p_k(x)$ , where  $p_i(x)$  has degree  $i$ .

One common choice is the Chebyshev polynomials. On  $[-1, 1]$ , these are defined by

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{j+1}(x) = 2xT_j(x) - T_{j-1}(x).$$

More generally, we can define them on an interval  $[a, b]$  by

$$p_i(x) = T_i\left(\frac{2x - (a + b)}{b - a}\right).$$

Repeat the above process with the scaled Chebyshev polynomials (here  $a = 0$  and  $b = 8$ ); i.e. using

$$\mathbf{A} = \begin{bmatrix} p_0(t_1) & p_1(t_1) & \cdots & p_k(t_1) \\ T_0(t_2) & p_1(t_2) & \cdots & p_k(t_2) \\ \vdots & \vdots & & \vdots \\ p_0(t_n) & p_1(t_n) & \cdots & p_k(t_n) \end{bmatrix}.$$

We can make a function to evaluate the Chebyshev polynomials on  $[a, b]$  as follows:<sup>1</sup>

```
def chebyshev_polynomial(j, x, a, b):
    return np.cos(j*np.arccos((2*x-(a+b))/(b-a)))
```

If we want to evaluate this for  $j = 3$  at all the  $t$  values we can do:

```
chebyshev_polynomial(3, t, 0, 8)
```

Make a plot with  $k = 5, 10, 15, 20, 25$ .

- (c) Explain why the plots should look the same if we were doing the computations exactly.
- (d) Look at the condition numbers of all of the matrices you use in the least squares problems. How does this explain why the plots with different polynomial families look different?

---

<sup>1</sup>It's not obvious how to get this formula, but you could prove it satisfies the recurrence formula! You can look at the wikipedia page for more info

**Problem 2.** Suppose  $\mathbf{A}$  is symmetric with eigenvalues  $\lambda_1, \dots, \lambda_n$  (so that  $|\lambda_1| > |\lambda_2| \geq \dots \geq |\lambda_n|$ ) and corresponding orthonormal eigenvectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$ .

Our analysis of the power-method involved writing the starting vector  $\mathbf{x}$  in terms of  $\mathbf{V}$ ; i.e. writing

$$\mathbf{x} = \mathbf{V} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

As long as  $|c_1| > 0$ , then we got convergence to  $\mathbf{v}_1$ .

Of course, in practice we don't know the eigenvectors or the  $c_i$ s. However, it turns out if we choose  $\mathbf{x}$  randomly, then  $c_1$  will never be zero (and in fact will never be that small).

- Suppose we have  $\mathbf{x}$  and  $\mathbf{V}$ . How we compute  $c_1$ ?
  - Make any  $5 \times 5$  symmetric matrix  $\mathbf{A}$  sample a length 5 vector  $\mathbf{x}$  whose entries are independent Gaussians. You can do this by using `np.random.randn(5)`.
  - Use numpy's `np.linalg.eigh` to compute its eigendecomposition. Use (a) to compute  $c_1$  and report it's value.
  - Repeat (b) 1000 times with a new  $\mathbf{x}$  each time. Make a histogram of the value of  $c_1$  over these trials.<sup>2</sup>
- Was  $c_1$  ever zero?
- In the above process, we needed to compute  $\mathbf{V}$  in order to find  $c_1$ . But this would be expensive for large matrices.

Explain why we do not need to compute  $\mathbf{V}$  in practice in order to use the power method to find  $\mathbf{v}_1$ .

**Problem 3.** Suppose  $\mathbf{A}$  is symmetric with eigenvalue decomposition  $\mathbf{V}\mathbf{\Lambda}\mathbf{V}^T$ , where  $\mathbf{\Lambda}$  is diagonal with entries  $\lambda_1, \dots, \lambda_n$ .

- Find the eigenvalue decomposition of:
  - $\mathbf{A}^k$
  - $\mathbf{A}^3 - 2\mathbf{A}$
  - $\mathbf{A}^{-1}$
  - $(\mathbf{A} + \lambda)^{-1}$
- What is the largest eigenvalue of  $(\mathbf{A} + \lambda)^{-1}$  in terms of the  $\lambda_i$ s?

---

<sup>2</sup>It turns out the distribution of  $c_1$  does not depend on how you generated the matrix  $\mathbf{A}$ ! This is because of something called the orthogonal invariance of the Gaussian distribution.

**Problem 4.** Suppose  $\mathbf{A}$  is symmetric with eigenvalues  $\lambda_1, \dots, \lambda_n$  (so that  $|\lambda_1| > |\lambda_2| > |\lambda_3| > \dots > |\lambda_n|$ ) and corresponding orthonormal eigenvectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$ .

Define  $\mathbf{B} = \mathbf{A}(\mathbf{I} - \mathbf{v}_1\mathbf{v}_1^\top)$ .

- (a) What are the eigenvalues of  $\mathbf{B}$ ?
- (b) Explain how to use the observation in (a) and the power-method to find  $\mathbf{v}_2$ .