Midterm (Practice Problems) Numerical Analysis Fall 2023

Name: ______________________ NetID:

Instructions

- Do not open the exam booklet until instructed.
- Clearly justify each step.
- Write only inside the indicated areas. Use scratch paper for scratch work.
- Circle your final answer

Comments

- The midterm will be 75 minutes and will cover all aspects of the course through the lecture on November 2.
- The problems on the exam will be similar to quiz problems, and will have between 5-10 problems (depending on the length).
- The problems are meant to be straightforward if you have a strong handle on the concepts. There may also be some computational problems.
- You should also study the quiz problems from this and last semester that aren't included in this document, as well as the lecture notes.

Problem 1. Define

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 3 \\ -2 \\ 4 \\ 1 \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

a) Compute $\mathbf{x}\mathbf{y}^\top\mathbf{z}$.

b) Compute $\mathbf{z}^\top\mathbf{y}\mathbf{x}^\top$.

Problem 2. A matrix $\mathbf{T}$ is said to be *checkered* if $[\mathbf{T}]_{i,j} = 0$ for all i, j where $|i - j|$ is odd.

Let $\mathbf{A}$ be a $n \times n$ checkered matrix and $\mathbf{D}$ a $n \times n$ diagonal matrix. Prove that $\mathbf{DA}$ is checkered.

Problem 3. Note that

$$\underbrace{\begin{bmatrix} 795/68 & 154/17 \\ 120/17 & 84/17 \\ 120/17 & 84/17 \\ 315/68 & 70/17 \end{bmatrix}}_{\mathbf{a}_1 \quad \mathbf{a}_2} \begin{bmatrix} -4/5 & 2/3 \\ 8/7 & -3/4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \underbrace{\begin{bmatrix} 13/4 & -2 & 479/6 \\ 3/4 & 0 & 25/2 \\ -14/3 & -2/4 & -1016/9 \\ 2/3 & -1 & 158/9 \end{bmatrix}}_{\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3} \begin{bmatrix} 14/3 \\ -2/5 \\ -1/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Compute the following:

a) $$\underbrace{\begin{bmatrix} 795/68 & 154/17 \\ 120/17 & 84/17 \\ 120/17 & 84/17 \\ 315/68 & 70/17 \end{bmatrix}}_{\mathbf{a}_1 \quad \mathbf{a}_2} \begin{bmatrix} (-4/5 + 2/3) \\ (8/7 - 3/4) \end{bmatrix},$$

b) $$\underbrace{\begin{bmatrix} 795/68 & 13/4 & -2 & 479/6 & 154/17 \\ 120/17 & 3/4 & 0 & 25/2 & 84/17 \\ 120/17 & -14/3 & -2/4 & -1016/9 & 84/17 \\ 315/68 & 2/3 & -1 & 158/9 & 70/17 \end{bmatrix}}_{\mathbf{a}_1 \quad \mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \quad \mathbf{a}_2} \begin{bmatrix} 4/5 & 8/3 \\ 14/3 & 0 \\ -2/5 & 0 \\ -1/5 & 0 \\ -8/7 & -3 \end{bmatrix}$$

a)

b)

Problem 4. Define

$$\mathbf{A} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

a) Compute $\|\mathbf{A}\|_\mathsf{F}$.

b) Compute $\|\mathbf{A}^\mathsf{T}\mathbf{A}\|_2$.

Problem 5. Let $\mathbb{F}$ denote some discrete set of numbers, and suppose that for some $\epsilon > 0$ the function $\mathrm{rd} : \mathbb{R} \to \mathbb{F}$ satisfies

$$|x - \mathrm{rd}(x)| < \epsilon |x|, \qquad \forall x \in \mathbb{R}.$$

Find the largest value of ϵ for which we can guarantee $\mathrm{rd}(10^5 + 1) \neq 10^5$.

For this value of ϵ, can we guarantee $\mathrm{rd}(10^{-6} - 10^{-10}) \neq 10^{-6}$?

Problem 6. Consider the following problem: Given a coefficients $a, c > 0$, solve $ax^2 - c = 0$.

Is this task well-conditioned? If so, prove it. If not, find an example demonstrating it is not.

Problem 7. Consider the following problem: Given a differentiable function $h : [-1, 1] \to \mathbb{R}$, return $h'(0)$.

Consider the following algorithm for this task: Given a function $h : [-1, 1] \to \mathbb{R}$, return

$$\frac{h(0.01) - h(0)}{0.01}.$$

Let's measure the distance between functions h and $\tilde{h}$ by $\|h - \tilde{h}\|_\infty = \max_{s \in [-1,1]} |h(s) - \tilde{h}(s)|$ and the distance between numbers with the regular absolute value.

a) Find an example which demonstrates that this algorithm is not forward stable.

b) Prove this algorithm is backwards stable.

b) Is this problem well-conditioned?

Problem 8. Obtain an LU factorization of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -3 \\ -2 & 0 & 5 \\ 5 & 2 & -10 \end{bmatrix}$$

Problem 9. Obtain a QR factorization of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 4 \\ 1 & -3 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 6 \end{bmatrix}$$

Problem 10. Define

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad \mathbf{U} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{V} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{W} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Note that

$$\operatorname{proj}_{\mathbf{U}}(\mathbf{x}) = \frac{1}{3} \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad \operatorname{proj}_{\mathbf{V}}(\mathbf{x}) = \frac{1}{3} \begin{bmatrix} 2 \\ 9 \\ 7 \\ 11 \end{bmatrix}$$

What is $\operatorname{proj}_{\mathbf{W}^{\perp}}(\mathbf{x})$?

Problem 11. Define

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 1 \\ 2 & -1 & -1 \\ 2 & 1 & 0 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 6 \\ 12 \\ -6 \\ -6 \end{bmatrix}.$$

a) Show $\mathbf{A}$ has orthogonal columns, and compute the norm of each column.

b) Find the solution $\mathbf{x}$ to the least squares problem $\min_{\mathbf{x}\in\mathbb{R}^3} \|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2^2$.

Hint: think about what part a) tells you about the structure of $\mathbf{A}^\top\mathbf{A}$.