Midterm — Numerical Analysis Fall 2023

Name: ____________________ NetID: | | | | | | | | | | | |

Do not open the exam booklet until instructed!

Instructions

- You can use one 8.5x11 inch sheet of notes.
- You can use a scientific calculator ($\leq$ 2 line display and no linear algebra functionality).
- Clearly justify each step.
- Write only inside the indicated areas; use scratch paper for scratch work.
- Write your answers in the given blank matrices/vectors areas when appropriate.
- Do not fold the corners of the papers.

Contents

Problem 1. a) Complete the following definition:

A matrix $\mathbf{A}$ is said to be *lower triangular* if:

b) Let $\mathbf{A}$ be a $n \times n$ lower triangular matrix and $\mathbf{D}$ a $n \times n$ diagonal matrix. Prove that $\mathbf{DA}$ is lower triangular.

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Problem 2. Suppose that $\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are vectors such that:

$$\begin{bmatrix} | & | \\ \mathbf{a}_1 & \mathbf{a}_2 \\ | & | \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -1 & \pi \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \begin{bmatrix} | & | & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} 14/3 \\ 12/5 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Compute the following:

$$\text{a) } \mathbf{x} = \begin{bmatrix} | & | \\ \mathbf{a}_1 & \mathbf{a}_2 \\ | & | \end{bmatrix} \begin{bmatrix} 5 \\ \pi - 1 \end{bmatrix}, \qquad \text{b) } \mathbf{y} = \begin{bmatrix} | & | & | & | & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{a}_1 & \mathbf{a}_2 \\ | & | & | & | & | \end{bmatrix} \begin{bmatrix} 14/3 \\ 12/5 \\ -2/3 \\ 4 \\ 2\pi \end{bmatrix}$$

a)

$\mathbf{x} =$ [___ , ___ , ___ , ___]

b)

$\mathbf{y} =$ [___ , ___ , ___ , ___]

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Problem 3. a) If $\mathbf{U\Sigma V}^\mathsf{T}$ is an SVD of an $n \times n$ matrix $\mathbf{A}$, what properties must $\mathbf{U}$, $\mathbf{\Sigma}$, and $\mathbf{V}$ have?

Define

$$\mathbf{A} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} -4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

b) Find an SVD of $\mathbf{A}$; $\mathbf{A} = \mathbf{U\Sigma V}^\mathsf{T}$. Note that you should specify $\mathbf{V}^\mathsf{T}$ not $\mathbf{V}$.

$$\mathbf{U} = \begin{bmatrix} & \\ & \end{bmatrix}, \quad \mathbf{\Sigma} = \begin{bmatrix} & \\ & \end{bmatrix}, \quad \mathbf{V}^\mathsf{T} = \begin{bmatrix} & \\ & \end{bmatrix}$$

c) What is $\max_{\mathbf{x} \in \mathbb{R}^2} \frac{\|\mathbf{Ax}\|_2}{\|\mathbf{x}\|_2}$?

d) What is $\|\mathbf{A}\|_\mathsf{F}$?

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Problem 4. Consider the following problem/task: Given an input of coefficients a and b, output a number x so that $ax + b = 0$.

a) Give a reasonable mathematical definition for the condition number of *this* problem; i.e. you should write an expression involving the specifics of this problem.

b) Is this task well-conditioned? If so, prove it. If not, find an example demonstrating it is not.

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Problem 5. Find a factorization $\mathbf{A} = \mathbf{LU}$, where $\mathbf{L}$ is lower triangular, and $\mathbf{U}$ is upper triangular.

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 4 \\ -1 & 5 & 3 \end{bmatrix}.$$

$$\mathbf{L} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

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Problem 6. Suppose we have run Gaussian elimination with partial pivoting on $\mathbf{A}$ to get:

$$\underbrace{\begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & 1/2 & 1 \end{bmatrix}}_{\mathbf{L}_2} \underbrace{\begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & 1 & \cdot \end{bmatrix}}_{\mathbf{P}_2} \underbrace{\begin{bmatrix} 1 & \cdot & \cdot \\ 1/2 & 1 & \cdot \\ 1/3 & \cdot & 1 \end{bmatrix}}_{\mathbf{L}_1} \underbrace{\begin{bmatrix} \cdot & 1 & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \end{bmatrix}}_{\mathbf{P}_1} \mathbf{A} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{bmatrix}$$

Write down a factorization $\mathbf{PA} = \mathbf{LU}$, where $\mathbf{P}$ is a permutation matrix, $\mathbf{L}$ is lower triangular, and $\mathbf{U}$ is upper triangular.

$\mathbf{P} = \begin{bmatrix} \; & \; & \; \\ \; & \; & \; \\ \; & \; & \; \end{bmatrix}$, $\mathbf{L} = \begin{bmatrix} \; & \; & \; \\ \; & \; & \; \\ \; & \; & \; \end{bmatrix}$, $\mathbf{U} = \begin{bmatrix} \; & \; & \; \\ \; & \; & \; \\ \; & \; & \; \end{bmatrix}$

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Problem 7. Find a factorization $\mathbf{A} = \mathbf{QR}$ where $\mathbf{Q}$ has orthonormal columns and $\mathbf{R}$ is upper triangular.

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & 4 \\ 0 & 1 & 1 \\ -3 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix}.$$
$$\quad \mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3$$

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$$\mathbf{Q} = \begin{bmatrix} & & \\ & & \\ & & \\ & & \end{bmatrix}, \qquad \mathbf{R} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

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Problem 8. Define

$$\mathbf{x} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{W} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ and note that: } \quad \mathrm{proj}_{\mathbf{W}}(\mathbf{x}) = \begin{bmatrix} 3 \\ 4 \\ 1 \\ 1 \end{bmatrix}$$

Now, define

$$\mathbf{y} = \mathbf{x} + 12 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

What is $\mathrm{proj}_{\mathbf{W}^\perp}(\mathbf{y})$, the projection of $\mathbf{y}$ onto the orthogonal compliment of $\mathbf{W}$?

$$\mathrm{proj}_{\mathbf{W}^\perp}(\mathbf{y}) = \begin{bmatrix} \text{-----} \\ \text{-----} \\ \text{-----} \\ \text{-----} \end{bmatrix}$$

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Problem 9. Define

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Find the solution $\mathbf{x}$ to the least squares problem $\min_{\mathbf{x}\in\mathbb{R}^2} \|\mathbf{b} - \mathbf{A}\mathbf{x}\|_2^2$.

$$\mathbf{x} = \begin{bmatrix} \\ \hline \end{bmatrix}$$

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