Homework 7 — Numerical Analysis Fall 2023

Instructions:

- Due 12/15 at 5:00pm on Gradescope.
- You must follow the submission policy in the syllabus
- This homework is worth 40 points (less than the usual 50 points)
- If this is your lowest homework grade, I will drop it (i.e. this homework is optional)

Problem 1. Spend at least two hours working on your project prior to the 20th. Answer the following:

(a) What is the current status of your project?

(b) What are the big tasks you have left to do before your project is done?

(c) What is your plan for completing the project in a timely manner?

Problem 2. Suppose $\mathbf{A}$ has SVD $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\mathsf{T}$ where

$$\mathbf{U} = \begin{bmatrix} | & | & & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \\ | & | & & | \end{bmatrix}, \qquad \mathbf{\Sigma} = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \\ | & | & & | \end{bmatrix}.$$

(a) Show that $\mathbf{v}_i$ is an eigenvector of $\mathbf{A}^\mathsf{T}\mathbf{A}$. What is the corresponding eigenvalue?

(b) Define the block matrix:

$$\mathbf{B} = \begin{bmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{A}^\mathsf{T} & \mathbf{0} \end{bmatrix}.$$

Show that

$$\mathbf{x} = \begin{bmatrix} \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix}$$

is an eigenvector of $\mathbf{B}$. What is the corresponding eigenvalue?

Problem 3. Let $f(x) = |x|$ and define $p_k(x)$ as the degree k polynomial interpolate to $f(x)$ at the $k+1$ Chebyshev nodes.

(a) For $k = 10$ make one plot with x vs $f(x)$ and x vs $p_k(x)$ and one plot over x vs $|f(x) - p_k(x)|$ with the vertical axis on a log-scale.

(b) Repeat this for $k = 40$

Problem 4. (a) Let $f(x) = \exp(-x)$ and define $p_k(x)$ as the degree k polynomial interpolate to $f(x)$ at the $k+1$ Chebyshev nodes.

Make a plot of k vs

$$\max_{x\in[-1,1]} |f(x) - p_k(x)|$$

for $k = 0, 1, 2, \ldots 20$. Put the y-axis on a log scale and label the axes/plot/etc.

To approximate

$$\max_{x\in[-1,1]} |f(x) - p_k(x)|$$

you can instead take the maximum over 1000 equally spaced points in $[-1, 1]$ and use this instead.

(b) Repeat this for $f(x) = 1/(1 + 16x^2)$ and $k = 0, 1, \ldots, 100$.

(c) Repeat this for $f(x) = |\sin(5x)|^3 = (\sin(5x)^2)^{3/2}$ and $k = 0, 1, \ldots, 100$, but put both axes on log-scales.

Add a line k vs $k^{-\nu}$, where ν is the largest value so that the $(\nu - 1)$-st derivative of $f(x)$ is continuous.

Problem 5. Let $f(x) = 1/(1 + 16x^2)$.

For any non-negative integer k, set $n = k^2 + 1$ and let $x_1, \ldots, x_n$ be n equally spaced points from -1 to 1 and let $q_k(x)$ be the degree k polynomial minimizing

$$\min_{\deg(q)=k} \sum_{i=1}^{n} (f(x_i) - q(x_i))^2.$$

On a log-y plot, plot the error

$$\max_{x\in[-1,1]} |f(x) - q_k(x)|$$

for $k = 0, 1, \ldots, 100$. Add to this plot the error of the Chebyshev interpolant that you computed in 3(b).