Homework 4 Numerical Analysis Fall 2023

Instructions:

- Due 10/27 at 5:00pm on Gradescope.
- You must follow the submission policy in the syllabus

Problem 1. In the last homework, we considered the following problem/task: You are given function $h : [-1, 1] \to \mathbb{R}$ and must return $\int_{-1}^{1} h(s)\mathrm{d}s$; i.e.

$$f(h) = \int_{-1}^{1} h(s)\mathrm{d}s.$$

Consider the following algorithm for this task:

$$\tilde{f}(h) = \sum_{i=0}^{100} \frac{1}{50} h(x_i), \qquad x_i = -1 + i/50.$$

(a) For each of the following inputs, compute the algorithm's output and compare it to the true solution $f(x)$.

input x	solution $f(x)$
$h(s) = 1$	2
$h(s) = s^2$	2/3
$h(s) = \sin(s)$	0

(b) Find an input for which the algorithm's output is very far from the true output. Explain why this is the case.

(c) Is this algorithms backwards stable? Justify your response.

Problem 2. For each $j = 1, 2, \dots, n-1$, define $\mathbf{L}_j \in \mathbb{R}^{n \times n}$ by

$$[\mathbf{L}]_{i,k} = \begin{cases} 1 & i = k \\ \ell_{i,j} & k = j \\ 0 & \text{o.w.} \end{cases}$$

For example:

$$\mathbf{L}_1 = \begin{bmatrix} 1 & & & & \\ \ell_{2,1} & 1 & & & \\ \ell_{3,1} & & 1 & & \\ \vdots & & & \ddots & \\ \ell_{n,1} & & & & 1 \end{bmatrix}, \quad \mathbf{L}_2 = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & \ell_{3,2} & 1 & & \\ & \vdots & & \ddots & \\ & \ell_{n,2} & & & 1 \end{bmatrix}, \dots \quad \mathbf{L}_{n-1} = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & \ell_{n,n-1} & 1 \end{bmatrix}$$

Verify that

$$(\mathbf{L}_{n-1}\cdots\mathbf{L}_2\mathbf{L}_1)^{-1} = \begin{bmatrix} 1 & & & & \\ -\ell_{2,1} & 1 & & & \\ -\ell_{3,1} & -\ell_{3,2} & 1 & & \\ \vdots & \vdots & & \ddots & \\ -\ell_{n,1} & -\ell_{n,2} & \cdots & -\ell_{n,n-1} & 1 \end{bmatrix}.$$

Hint: It suffices to check a certain product of matrices is the identity. Write down this product, and then compute it.

Problem 3. Suppose

$$\mathbf{A} = \begin{bmatrix} 0 & 3 & -2 & 4 \\ 4 & -3 & 2 & 1 \\ 1 & 2 & 3 & -1 \\ 2 & 1 & 1 & 3 \end{bmatrix}$$

(a) Perform PLU factorization, using the row with the largest leading entry as the pivot row, to obtain a factorization $\mathbf{L}^{(3)}\mathbf{P}^{(3)}\mathbf{L}^{(2)}\mathbf{P}^{(2)}\mathbf{L}^{(1)}\mathbf{P}^{(1)}\mathbf{A} = \mathbf{U}$. Write each of the $\mathbf{L}^{(i)}$, $\mathbf{P}^{(i)}$, and $\mathbf{U}$ as you go, along with the current state of the matrix after applying each factor.

You should compute the factors exactly (i.e. using fractions), but you do not need to show you work when evaluating matrix-matrix products. It is recommended you use wolfram alpha, Mathematica, sympy, or some other symbolic math tool to assist you.

(b) Use (a) to find a factorization $\mathbf{PA} = \mathbf{LU}$.

(c) Perform regular LU factorization (without pivoting) on $\mathbf{PA}$. Show the row operation matrices you use along the way.

(d) How do the steps you take in (a) and (c) compare?

Problem 4. Let

$$\mathbf{Q} = \begin{bmatrix} | & | & & | \\ \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_k \\ | & | & & | \end{bmatrix}.$$

(a) Suppose $\mathbf{Q}^\top\mathbf{Q} = \mathbf{I}$. What is $\mathbf{q}_i^\top\mathbf{q}_j$ for each $i, j \in \{1, 2, \ldots, k\}$?

(b) Suppose $\mathbf{x}$ is in the span of $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_k\}$. Show that $\mathbf{x} = \mathbf{Qc}$ for some vector $\mathbf{c} \in \mathbb{R}^k$.

(c) Suppose we know $\mathbf{x} = c_1\mathbf{q}_1 + c_2\mathbf{q}_2 + \cdots + c_k\mathbf{q}_k$, but we do not know the coefficients $c_1, c_2 \ldots, c_k$. Explain how you can obtain these coefficients from $\mathbf{Q}$ and $\mathbf{x}$.