Homework 2: Mathematical Statistics (MATH-UA 234)

Due 09/22 at the beginning of class on Gradescope

Problem 1. *Solve each of the following:*

(a) *Let X be any random variable with* $\mathbb{E}[X^4] < \infty$. *Show that* $\mathbb{E}[X^4] \geq \mathbb{E}[X^2]^2$.

(b) *Suppose* $X \sim \operatorname{Exp}(1)$. *Then, as in Example 3.30, the moment generating function is* $\psi_X(t) = 1/(1-t)$. *Use the moment generating function to find* $\mathbb{E}[X^k]$, *for integer* $k \geq 0$.

(c) *Let* $X \sim \operatorname{Exp}(1)$ *and let* $Y = \cos(X)$. *Find* $\mathbb{E}[Y]$.

(d) *Let X, Y be random variables. Show that* $\operatorname{Co}\mathbb{V}[X,Y] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$.

Problem 2. *Suppose X and Y are random variables with joint probability mass function,*

$$f_{X,Y}(x,y) = \mathbb{P}[X = x, Y = y] = \begin{cases} .1 & X = -1, Y = 1 \\ .3 & X = -1, Y = -1 \\ .2 & X = 1, Y = 1 \\ .4 & X = 1, Y = -1 \end{cases}$$

(a) *Compute the marginal probability mass function,* $f_X(x) = \mathbb{P}[X = x]$.

(b) *Compute* $f(y) = \mathbb{E}[X|Y = y]$.

(c) *Compute* $\mathbb{E}[X]$ *using the marginal pmf and then using the the law of iterated expectation. Do the results agree?*

Problem 3 (Wasserman 3.4 (statistics of a random walk))**.** *A particle starts at the origin of the real line and moves along the line in jumps of one unit. For each jump the probability is p that the particle will jump one unit to the left and the probability is* $1-p$ *that the particle will jump one unit to the right. Let* X_n *be the position of the particle after n jumps. Find* $\mathbb{E}[X_n]$ *and* $\mathbb{V}[X_n]$. *(This is known as a random walk.)*

Problem 4 (Wasserman 3.15 (variance of a mixture))**.** *Let*

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{3}(x+y) & 0 \leq x \leq 1, 0 \leq y \leq 2 \\ 0 & \textit{otherwise.} \end{cases}$$

Find $\mathbb{V}[2X - 3Y + 8]$.

Problem 5 (Wasserman 5.4 (convergence))**.**

Let $X_1, X_2, \ldots$ *be a sequence of random variables such that*

$$\mathbb{P}[X_n = 1/n] = 1 - 1/n^2, \quad \mathbb{P}[X_n = n] = 1/n^2.$$

(a) *Does* X_n *converge in probability to any random variable? If so, prove this. If no such variable exists, explain why not.*

(b) *Does* X_n *converge in quadratic mean? If so, prove this. If no such variable exists, explain why not.*

Problem 6. *Describe an instance of one of the probability concepts we've seen recently in the course which you noticed in a different part of your life (e.g. in other classes, on the subway, at the park, in the dorm, etc.).*

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problems with a textbook reference are based on, but not identical to, the given reference