Announcements

- HW4: Typo in V_{boot} (now fixed)
- Quiz 6 (11/22) will be optional
- Cheating: HW must be written alone
  Quizzes no talking, looking at other people, etc.

Parameter Inference

$\mathcal{F} = \{ f_\theta(x) : \theta \in \Theta \}$ parametric model

Ex. $\mathcal{F} = \left\{ \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(X-\mu)^2}{2\sigma^2}\right), (\mu,\sigma) \in \mathbb{R} \times \mathbb{R}_{>0} \right\}$

$\theta = (\mu, \sigma)$ $\Theta = \mathbb{R} \times \mathbb{R}_{>0}$

Goal: Learn θ from data

Def. A parameter of interest is some quantity (depend on θ) that we care about.

Ex. $\mu = T(\theta)$

Ex. $\tau = 1 - \Phi\left(\frac{1-\mu}{\sigma}\right)$, $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz$

Def. The j-th moment of a RV X is $\mathbb{E}[X^j]$.

Def. The j-th moment of a distribution F_θ is

$$\alpha_j = \alpha_j(\theta) = \int x^j \, dF_\theta(x) = \mathbb{E}[Z^j]$$

$Z \sim F$

Def. The j-th sample moment of iid data $X_1, \ldots, X_n$ is $\hat{\alpha}_j = \frac{1}{n} \sum_{i=1}^{n} X_i^j$

Idea: find the value of θ that gives the same moments as what are observed in the data.

Def. Let $\mathcal{F} = \{f_\theta(x) : \theta \in \Theta\}$, where $\Theta \in \mathbb{R}^k$.
The method of moments estimator is the parameter $\hat{\theta}_n \in \mathbb{R}^k$ such that

$(\hat{\theta}_{n,1}, \hat{\theta}_{n,2}, \ldots, \hat{\theta}_{n,k})$

$$\hat{\alpha}_1 = \alpha_1(\hat{\theta}_n) = \alpha_1(\hat{\theta}_{n,1}, \ldots, \hat{\theta}_{n,k})$$
$$\hat{\alpha}_2 = \alpha_2(\hat{\theta}_n) = \alpha_2(\hat{\theta}_{n,1}, \ldots, \hat{\theta}_{n,k})$$
$$\vdots$$
$$\hat{\alpha}_k = \alpha_k(\hat{\theta}_n) = \alpha_k(\hat{\theta}_{n,1}, \ldots, \hat{\theta}_{n,k})$$

This is k equations and k unknowns.

Ex. $\mathcal{F} = \{\text{Bernoulli with param } p\}$

$\theta = p \quad , \quad \Theta = (0,1)$

$\hat{\alpha}_1 = \frac{1}{n}\sum_{i=1}^{n} X_i \quad , \quad \alpha_1(\theta) = \mathbb{E}[\overset{X\sim \text{Ber}(p)}{X}] = 1\cdot p + 0\cdot(1-p) = p$

solve : $\hat{p}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ for $\hat{p}$

Ex. $\mathcal{F} = \{F_{\mu,\sigma^2} : \mu = \int x \, dF(x),\ \sigma^2 = \int (x-\mu)^2 dF(x),\ \mu \in \mathbb{R},\ \sigma^2 > 0\}$

$\hat{\alpha}_1 = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad \alpha_1 = \mu$

$\hat{\alpha}_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 \qquad \alpha_2 = \mathbb{E}[\overset{X\sim F_\theta}{X^2}] = \mathbb{V}[X] + \mathbb{E}[X]^2 = \sigma^2 + \mu^2$

solve: $\hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

$\hat{\mu}_n^2 + \hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$

$\Rightarrow \quad \hat{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

$\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2 = \frac{1}{n}\sum_{i=1}^{n} (X_i - \bar{X}_n)^2$

Theorem

Let $\hat{\theta}_n$ be the method of moments estimator. Under appropriate conditions:

1. $\mathbb{P}[\hat{\theta}_n \text{ exists}] \to 1$

2. $\hat{\theta}_n \xrightarrow{P} \theta$ (consistent)

3. $\sqrt{n}(\hat{\theta}_n - \theta) \to N(0, \Sigma)$

   where $\Sigma = g\, \mathbb{E}[YY^T]\, g^T$,

   $Y = (X, X^2, \ldots, X^k)^T$, $g = (g_1, \ldots, g_k)$

   $$g_j = \frac{\partial \alpha_j^{-1}(\theta)}{\partial \theta}$$