Announcements

- HW 2 due Thurs
  - Tag questions properly, make sure submission is readable, etc.
  - On question regarding real life, try to think of examples you actually come across..
- Quiz 2 in one week

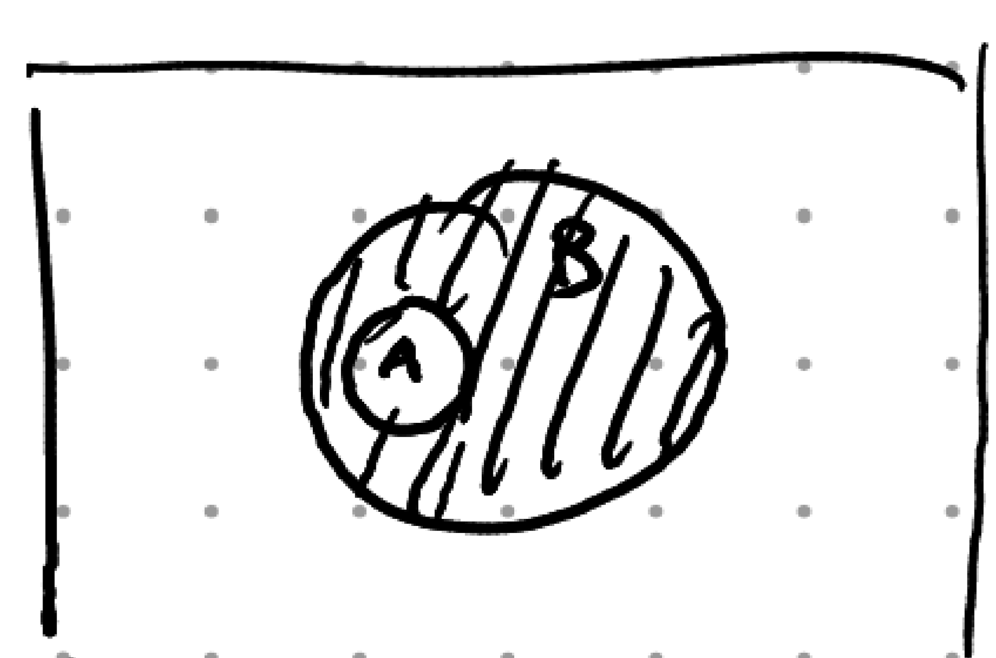
- $A \subseteq B \implies \mathbb{P}[A] \leq \mathbb{P}[B]$

$B = (B \cap A) \cup (B \cap A^c)$ $\quad$ $B \cap A$, $B \cap A^c$ disjoint

$$\mathbb{P}[B] = \mathbb{P}[B \cap A] + \mathbb{P}[B \cap A^c]$$
$$= \mathbb{P}[A] + \underbrace{\mathbb{P}[B \cap A^c]}_{\geq 0}$$
$$\geq \mathbb{P}[A]$$

- $A \subsetneq B \overset{?}{\implies} \mathbb{P}[A] < \mathbb{P}[B]$

i.e. $\mathbb{P}[B \cap A^c] > 0$?



$A = \{1\}$, $B = \{1, 3, 5\}$

all sides of dice labeled "1".

Convergence of RVs

$X_n \xrightarrow{d} X$ if

$\lim_{n \to \infty} F_n(t) = F(t) \quad \forall t$ where F is continuous

$F_n(t) = F_{X_n}(t)$, $F(t) = F_X(t)$

CLT

$X_1, \dots, X_n$ iid, $\mathbb{E} X_i = \mu$, $\mathbb{V}[X_i] = \sigma^2$

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

- $\mathbb{E}[\bar{X}_n] = \mu \qquad \mathbb{V}[\bar{X}_n] = \frac{\sigma^2}{n}$

- $Z_n = \dfrac{\bar{X}_n - \mu}{\sqrt{\mathbb{V}\,\bar{X}}} = \dfrac{\sqrt{n}\,(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} N(0,1)$

Theorem (Berry–Esseen)

Suppose $\mathbb{E}[|X_1|^3] < \infty$ Then

$$\sup_z \left| \mathbb{P}[Z_n \le z] - \Phi(z) \right| \le \frac{33}{4} \frac{\mathbb{E}[|X_1 - \mu|^3]}{\sqrt{n}\,\sigma^2}$$

$$S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2 \quad \text{Bessel's correction}$$

unbiased (sample variance)

WLOG: $\mathbb{E}[X_i] = 0 \Rightarrow \mathbb{E}[X_i^2] = \mathbb{V}[X_i] = \sigma^2$

$$\mathbb{E}[S_n^2] = \frac{1}{n-1} \sum_{i=1}^{n} \mathbb{E}[(X_i - \bar{X}_n)^2]$$

$$\mathbb{E}[(X_i - \bar{X}_n)^2] = \mathbb{E}[X_i^2] - 2\mathbb{E}[X_i \bar{X}_n] + \mathbb{E}[\bar{X}_n^2]$$

$$= \mathbb{E}[X_i^2] + \frac{\mathbb{E}[X_i^2]}{n} - \frac{2}{n}\sum_{j=1}^{n} \mathbb{E}[X_i X_j]$$

$$= \mathbb{E}[X_i^2]\left(1 + \frac{1}{n}\right) - \frac{2}{n}\mathbb{E}[X_i^2]$$

$$= \mathbb{E}[X^2]\left(1 - \frac{1}{n}\right)$$

$$= \sigma^2\left(\frac{n-1}{n}\right)$$

CLT (sample)

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{S_n} \xrightarrow{d} N(0,1)$$

CLT (high dimension)

$\vec{X}_1, \dots, \vec{X}_n \in \mathbb{R}^k$ iid random vectors with mean $\vec{\mu}$ and covariance matrix Σ. Let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} \vec{X}_i$. Then

$$\sqrt{n}\,(\bar{X}-\mu) \xrightarrow{d} N(0,\Sigma).$$

$$f_{\vec{X}}(\vec{x}) = \frac{1}{\sqrt{\det(2\pi\Sigma)}} \exp\left(-\frac{x^T\Sigma^{-1}x}{2}\right)$$

$$\vec{X} = \begin{bmatrix} X^{(1)} \\ X^{(2)} \\ \vdots \\ X^{(k)} \end{bmatrix} \qquad \vec{\mu} = \begin{bmatrix} \mu^{(1)} \\ \mu^{(2)} \\ \vdots \\ \mu^{(k)} \end{bmatrix} = \begin{bmatrix} \mathbb{E}[X^{(1)}] \\ \mathbb{E}[X^{(2)}] \\ \vdots \\ \mathbb{E}[X^{(k)}] \end{bmatrix} = \mathbb{E}[\vec{X}]$$

$$\Sigma = \begin{bmatrix} \mathbb{E}\left[(X^{(1)}-\mu^{(1)})(X_i^{(1)}-\mu^{(1)})\right] & \cdots & \mathbb{E}\left[(X^{(1)}-\mu^{(1)})(X^{(k)}-\mu^{(k)})\right] \\ \vdots & & \vdots \\ \mathbb{E}\left[(X^{(k)}-\mu^{(k)})(X^{(1)}-\mu^{(1)})\right] & \cdots & \mathbb{E}\left[(X^{(k)}-\mu^{(k)})(X^{(k)}-\mu^{(k)})\right] \end{bmatrix}$$

$$= \mathbb{E}\left[(\vec{X}_i-\vec{\mu})(\vec{X}_i-\vec{\mu})^T\right]$$