Announcements

- Recitation Friday
- HW 1 due Thurs
- HW 2 due next Thurs (9/27)

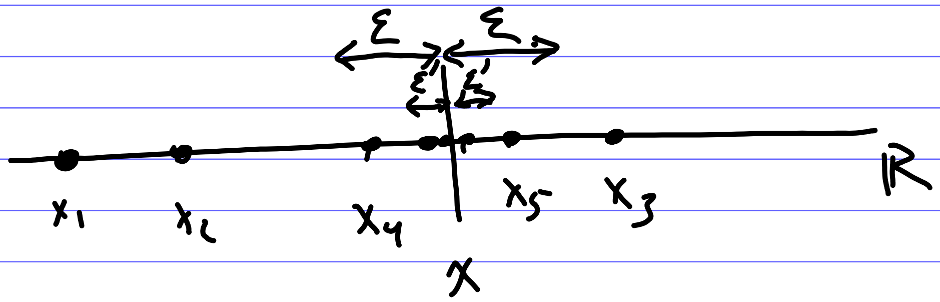
Convergence of sequence

$X_1, X_2, X_3, \ldots$

we say $X_n \to X$ if

$\forall \varepsilon > 0 \quad \exists N$ st $\forall n \geq N$

$$|X_n - X| < \varepsilon$$



Ex.

$X_n \sim \text{Unif}(0, \frac{1}{n})$ X_i independent

$X_n = (-1)^n X \quad , X \sim N(0,1)$

$$X_n = \begin{cases} \frac{1}{n} & \text{w.p. } 1 - \frac{1}{n^2} \\ n & \text{w.p. } \frac{1}{n^2} \end{cases}$$

Almost sure convergence

$X_n \to X$ almost surely if

$$\mathbb{P}\left[\{\omega \in \Omega : X_n(\omega) \to X(\omega)\}\right] = 1$$

Convergence in quadratic mean (L_2)

$X_n \to X$ in quadratic mean if

$$\mathbb{E}\left[(X_n - X)^2\right] \to 0$$

Convergence in probability

$X_n \to X$ in probability if $\forall \varepsilon > 0$

$$\mathbb{P}\left[\underbrace{|X_n - X| > \varepsilon}_{*}\right] \to 0$$

$$\mathbb{P}\left[\{\omega \in \Omega : |X_n(\omega) - X(\omega)| > \varepsilon\}\right]$$

Convergence in distribution

$X_n \to X$ in distribution if

$F_n(t) \to F(t)$ $\forall t$ where

F is continuous.

Theorem

$$\begin{tikzcd}
\boxed{\text{a.s.}} \arrow[dr] & & \\
& \boxed{\text{prob}} \arrow[r] & \boxed{\text{dist}} \\
\boxed{L_2} \arrow[ur] & &
\end{tikzcd}$$

Theorem

(a) $X_n \xrightarrow{P} X,\ Y_n \xrightarrow{P} Y \Rightarrow X_n + Y_n \xrightarrow{P} X + Y$

(d) $\Rightarrow X_n Y_n \xrightarrow{P} XY$

(b) $X_n \xrightarrow{qm} X,\ Y_n \xrightarrow{qm} Y \Rightarrow X_n + Y_n \xrightarrow{qm} X + Y$

(c) $X_n \xrightarrow{d} X,\ Y_n \xrightarrow{d} c \Rightarrow X_n + c \xrightarrow{d} X + c$

(e) $\Rightarrow cX_n \xrightarrow{d} cX$

(f) $X_n \xrightarrow{P} X \Rightarrow g(X_n) \xrightarrow{P} g(X)$

g cont.

(g) $X_n \xrightarrow{d} X \Rightarrow g(X_n) \xrightarrow{d} g(X)$

Ex. $X_1, X_2, \ldots, X_n$ independent identically distributed $\to$ iid

$\mathbb{E}[X_i] = \mu$

$\mathbb{V}[X_i] = \sigma^2 \qquad \left(\mathbb{V}[X_i] = \mathbb{E}\left[(X_i - \mu)^2\right]\right)$

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

What is $\mathbb{E}[\bar{X}_n]$, $\mathbb{V}[\bar{X}_n]$?

$$\begin{aligned}
\mathbb{E}[\bar{X}_n] &= \mathbb{E}\left[\frac{1}{n}(X_1 + \cdots X_n)\right] \\
&= \frac{1}{n}\mathbb{E}[X_1 + \cdots + X_n] \\
&= \frac{1}{n}\left(\mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n]\right) \\
&= \frac{1}{n}(\underbrace{\mu + \mu + \cdots \mu}_{n \text{ terms}}) \\
&= \mu
\end{aligned}$$

$$\begin{aligned}
\mathbb{V}[\bar{X}_n] &= \mathbb{V}\left[\frac{1}{n}(X_1 + \cdots + X_n)\right] \\
&= \frac{1}{n^2}\mathbb{V}[X_1 + \cdots + X_n] \\
&\overset{\text{independence!}}{=} \frac{1}{n^2}\left(\mathbb{V}[X_1] + \cdots + \mathbb{V}[X_n]\right) \\
&= \frac{1}{n^2}(\sigma^2 + \cdots \sigma^2) \\
&= \frac{\sigma^2}{n}
\end{aligned}$$

Law of Large Numbers

Th. (weak LLN)

$X_1, X_2, \dots, X_n$ are iid

Let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

Then $\bar{X}_n \xrightarrow{P} \mathbb{E}[X_i]$

Central Limit Theorem

Theorem (CLT)

$X_1, \dots, X_n$ iid $\quad \mathbb{E}[X_i] = \mu$

$\mathbb{V}[X_i] = \sigma^2$

$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

$$Z_n = \frac{\bar{X}_n - \mu}{\sqrt{\mathbb{V}[\bar{X}_n]}} = \sqrt{n}\left(\frac{\bar{X}_n - \mu}{\sigma}\right)$$

$Z_n \xrightarrow{d} Z, \quad Z \sim N(0,1)$

$$\lim_{n\to\infty} \underbrace{\mathbb{P}[Z_n \le z]}_{F_n(z)} \to \underbrace{\int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx}_{F(x)}$$

$$Z_n \approx N(0,1)$$

$$\bar{X}_n \approx N\left(\mu, \frac{\sigma^2}{n}\right)$$