Announcements

- Quiz 1 Tuesday, HW 1 Thursday
- OH M/T 6-7, Th 10-11
- No recitation this week
- Check you have 6/6 on Intro survey

Recap

- CDF : $F_X(x) = \mathbb{P}[X \leq x]$ (always)
- pmf : $f_X(x) = \mathbb{P}[X = x]$ (discrete)
- pdf : $f_X(x) = F_X'(x)$ (continuous)

$$= \mathbb{P}[X \in [x, x+dx]] / dx$$

Important RVs

- Unif(a, b)

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{o.w} \end{cases}$$

- $N(\mu, \sigma^2) = \dfrac{1}{\sigma\sqrt{2\pi}} \exp\left(-\dfrac{1}{2\sigma^2}(x-\mu)^2\right)$

- Gamma, Beta, Chi-squared, Exponential

Bivariate Distributions

- Joint CDF: $F_{X,Y}(x,y) = \mathbb{P}[X \leq x, Y \leq y]$

  - PDF: $\mathbb{P}[(X,Y) \in A] = \iint_A f_{X,Y}(x,y)\,dx\,dy$
  - PMF: $f_{X,Y}(x,y) = \mathbb{P}[X = x \text{ and } Y = y]$

Marginal Distributions

If X, Y have JMF $f_{X,Y}$, the marginal mass function for X is

$$f_X(x) = \mathbb{P}[X = x] = \sum_y \mathbb{P}[X = x, Y = y] = \sum_y f_{X,Y}(x,y)$$

marginal density

$$f_X(x) = \int f_{X,Y}(x,y)\,dy$$

Independent RVs

RVs X, Y are independent if

$$\mathbb{P}[X \in A \text{ and } Y \in B] = \mathbb{P}[X \in A]\,\mathbb{P}[Y \in B] \quad \forall A, B$$

RVs X, Y independent $\iff f_{X,Y}(x,y) = f_X(x) f_Y(y)$

Conditional Distributions

The conditional PMF is

$$f_{X|Y=y}(x,y) = \mathbb{P}[X=x \mid Y=y] = \frac{\mathbb{P}[X=x, Y=y]}{\mathbb{P}[Y=y]} = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

conditional density

$$f_{X|Y=y}(x,y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Transformation of RVs

Ex: $f_X(x) = e^{-x}$, $x > 0$, 0 $x \le 0$

- what is $F_X(x)$?

$$F_X(x) = \int_0^x f_X(t)\,dt = \left[-e^{-t}\right]_0^x = 1 - e^{-x}$$

- $Y = \log(X)$, what is $F_Y(y)$?

$$\begin{aligned} F_Y(y) = \mathbb{P}[Y \le y] &= \mathbb{P}[\log X \le y] \\ &= \mathbb{P}[X \le e^y] \\ &= F_X(e^y) = 1 - e^{-e^y} \end{aligned}$$

Expectation

The <u>expected value</u> or <u>mean</u> of X is

$$\mu = \mathbb{E}[X] = \int x f_X(x)\,dx \qquad \left(\sum_i x_i \mathbb{P}[X = x_i] \right)$$

$$\mathbb{E}[r(X)] = \int r(x) f_X(x)\,dx$$

Ex. What is expected value X, $f_X(x) = \exp(-x)$

<u>Indicator</u>

$$I_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

$$\mathbb{E}[I_A] = \int I_A(x) f_X(x)\,dx = \int_A f_X(x)\,dx = \mathbb{P}[X \in A]$$

<u>Properties</u>

- $\mathbb{E}[aX + bY] = a\mathbb{E}X + b\mathbb{E}Y$ (linearity)
- if X, Y independent $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$

<u>Conditional expectation</u>

$$\mathbb{E}[X \mid Y = y] = \int x f_{X|Y=y}(x)\,dx \qquad (\text{function of } y)$$

$\mathbb{E}[X \mid Y]$ random variable whose value is $\mathbb{E}[X \mid Y = y]$ when $Y = y$.

Iterated Expectation

For RVs X, Y,

$$\mathbb{E}\left[\mathbb{E}[X|Y]\right] = \mathbb{E}[X]$$

$$\mathbb{E}\left[\mathbb{E}[r(X,Y)|Y]\right] = \mathbb{E}[r(X,Y)]$$

Pf. $\mathbb{E}[\overbrace{\mathbb{E}[X|Y]}^{\text{has value } \mathbb{E}[X|Y=y] \text{ when } Y=y}] = \sum_y \mathbb{E}[X|Y=y]\,\mathbb{P}[Y=y]$

$$= \sum_y \left(\sum_x x\,\mathbb{P}[X=x|Y=y]\right)\mathbb{P}[Y=y]$$

$$= \sum_x x \sum_y \mathbb{P}[X=x|Y=y]\,\mathbb{P}[Y=y]$$

$$= \sum_x x\,\mathbb{P}[X=x]$$

$$= \mathbb{E}[X]$$

Law of total Expectation

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X|A_i]\,\mathbb{P}[A_i] \quad (A_i \text{ partition of } \Omega)$$

Moments

The k^{th} moment of X is $\mathbb{E}[X^k]$

The k^{th} centered moment of X is $\mathbb{E}[(X-\mathbb{E}X)^k]$

- Variance = centered 2^{nd} moment $(\mathbb{V}[X])$
- std. deviation = $\sigma_X = \sqrt{\mathbb{V}[X]}$ (same units as X)

Moment generating fxn

$$\psi_X(t) = \mathbb{E}[e^{tX}]$$

$$\left(e^a = 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \dots\right)$$

$$\mathbb{E}[e^{tX}] = \mathbb{E}\left[1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \dots\right]$$

$$= 1 + t\mathbb{E}[X] + \frac{t^2}{2!}\mathbb{E}[X^2] + \frac{t^3}{3!}\mathbb{E}[X^3] + \dots$$

- How can I get $\mathbb{E}[X^k]$ from $\psi_X(t)$?

Properties of Variance

- $\mathbb{V}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

  $\mathbb{E}[(X-\mu)^2] = \mathbb{E}[X^2 - 2\mu X + \mu^2] = \mathbb{E}[X^2] - 2\mu\mathbb{E}X + \mathbb{E}[X]^2$

  $= \mathbb{E}[X^2] - \mathbb{E}[X]^2$

- $\mathbb{V}[aX+b] = a^2\mathbb{V}[X]$

- if X, Y are independent,

  $\mathbb{V}[aX+bY] = a^2\mathbb{V}[X] + b^2\mathbb{V}[Y]$

Covariance

$$\mathrm{Co}\mathbb{V}(X,Y) = \mathbb{E}\left[(X-\mathbb{E}X)(Y-\mathbb{E}Y)\right]$$

correlation $\rho[X,Y] = \dfrac{\mathrm{Co}\mathbb{V}[X,Y]}{\sqrt{\mathbb{V}[X]\,\mathbb{V}[Y]}}$ (normalized covariance)

- $\mathrm{Co}\mathbb{V}[X,Y] = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$
- $\mathbb{V}[X+Y] = \mathbb{V}[X] + \mathbb{V}[Y] + 2\,\mathrm{Co}\mathbb{V}[X,Y]$