Random Variables

A random variable is a map from the sample space to reals

Ex. dice
$\Omega = \{1, 2, 3, 4, 5, 6\}$
$X(\omega) = \omega \quad \forall \omega \in \Omega$

Ex. flip coin 5x $\quad \Omega = \{(c_1 c_2 c_3 c_4 c_5) : c_i \in \{H, T\}\}$
$X(\omega) = \#$ heads
$X(HTTHH) = 3$

$\uparrow$
$\boxed{9/01}$

Announcements

- Intro survey today
- HW1 next Tuesday
- Quiz 1 next Tuesday

Ω = sample space (set of all outcomes of experiment)

- $\{1, 2, 3, 4, 5, 6\}$
- $\{(H,H), (H,T), (T,H), (T,T)\}$

event = subset of Ω (set of outcomes of experiment)

- $\{1, 3, 5\}$
- $\emptyset, \Omega$

probability distribution / measure

- $\mathbb{P}[A] \geq 0 \quad \forall$ event A
- $\mathbb{P}[\Omega] = 1$
- $\mathbb{P}[\bigcup_{i=1}^{\infty} A_i] = \sum_{i=1}^{\infty} \mathbb{P}[A_i]$ if $A_1, A_2, \ldots$ disjoint

independent events

$\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B]$

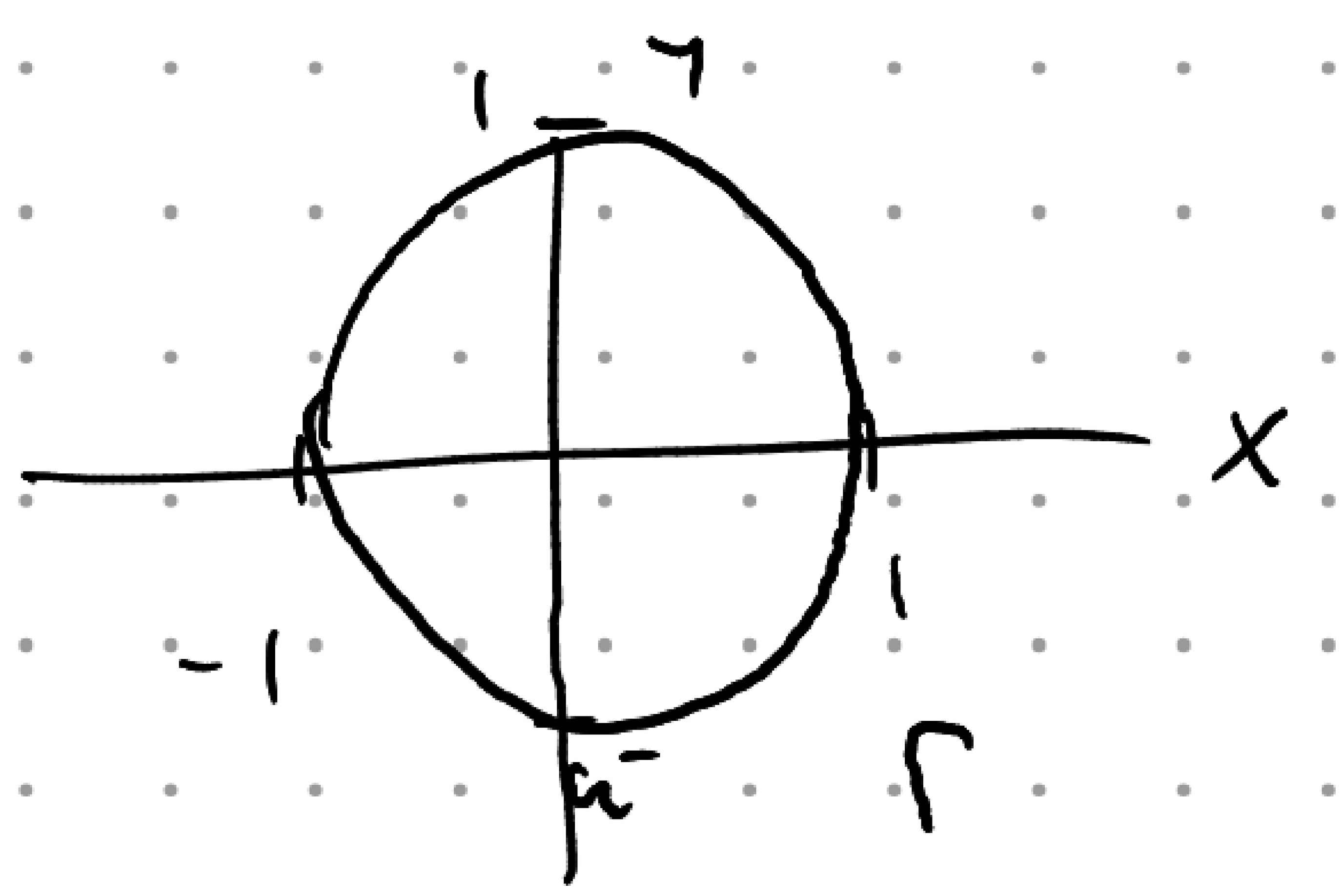
conditional probability

$\mathbb{P}[A \mid B] = \mathbb{P}[A \cap B] / \mathbb{P}[B]$

RV : map from $\Omega \to \mathbb{R}$

- $X(\omega)$ = # heads in ω
- $X(\omega)$ = payoff for roulette roll

Ex. Let $\Omega = \{(x,y) : x^2 + y^2 \leq 1\}$



Suppose we draw a point at random from Ω

Examples of RVs:

- $D(\omega) = \sqrt{x^2+y^2}$ (distance from $(0,0)$)
- $B(\omega) = \begin{cases} 1 & \text{if } D(\omega) \leq \frac{1}{10} \\ 0 & \text{otherwise} \end{cases}$ (did I get bullseye?)

Distribution functions

Let X be a RV.

If X is discrete, we can consider the <u>probability mass function</u> (pmf)

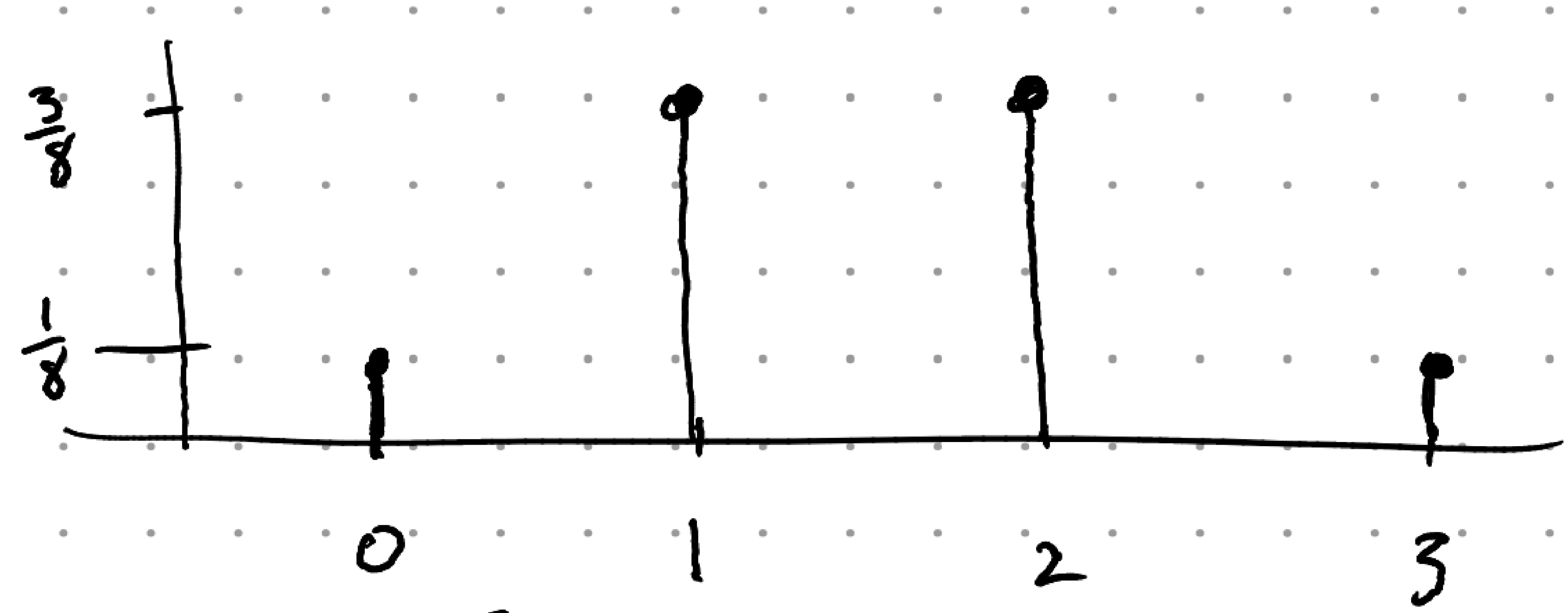
$$f_X(x) = \mathbb{P}[\underbrace{X = x}_{\text{shorthand for this event}}] = \mathbb{P}[\underbrace{\{\omega : X(\omega) = x\}}]$$

Ex. $\Omega = \{(\omega_1, \omega_2, \omega_3) : \omega_i \in \{H, T\}\}$

$X = \#$ heads

$\mathbb{P}[\{\omega \in \Omega : \omega_i = H\}] = \frac{1}{2} \quad i = 1, 2, 3$

i.e. fair coin



$$f_X(x) = \begin{cases} 1/8 & x = 0 \\ 3/8 & x = 1 \\ 3/8 & x = 2 \\ 1/8 & x = 3 \end{cases}$$

Ex. $\Omega = \{\omega : \omega \in [0,1]\} = [0,1]$

$\mathbb{P}$ = "all points equally likely"

$X(\omega) = \omega$

$\mathbb{P}[(a,b)] = b-a$ (ideally)

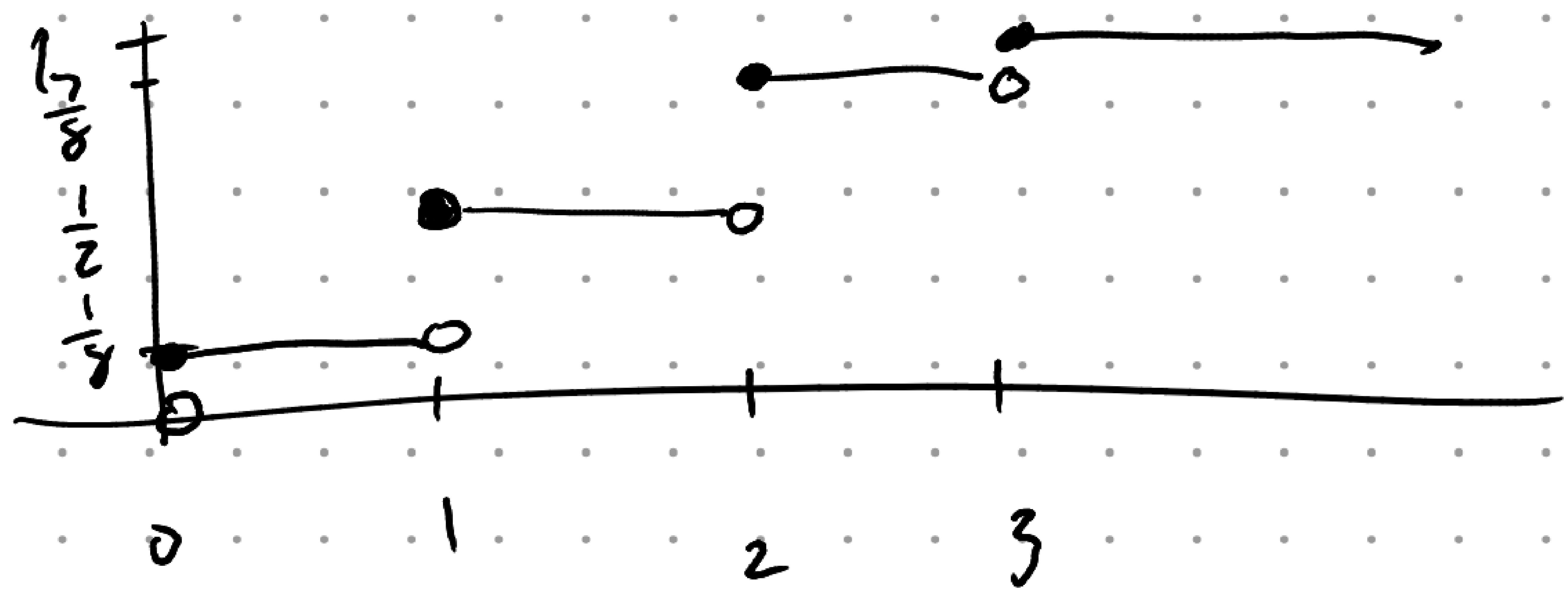
but then $\mathbb{P}[\{\omega\}] = 0$ ... so $\mathbb{P}[X=x]=0$ $\forall x \in [0,1]$

pmf doesn't make sense here..

<u>Cumulative distribution function</u> (CDF)

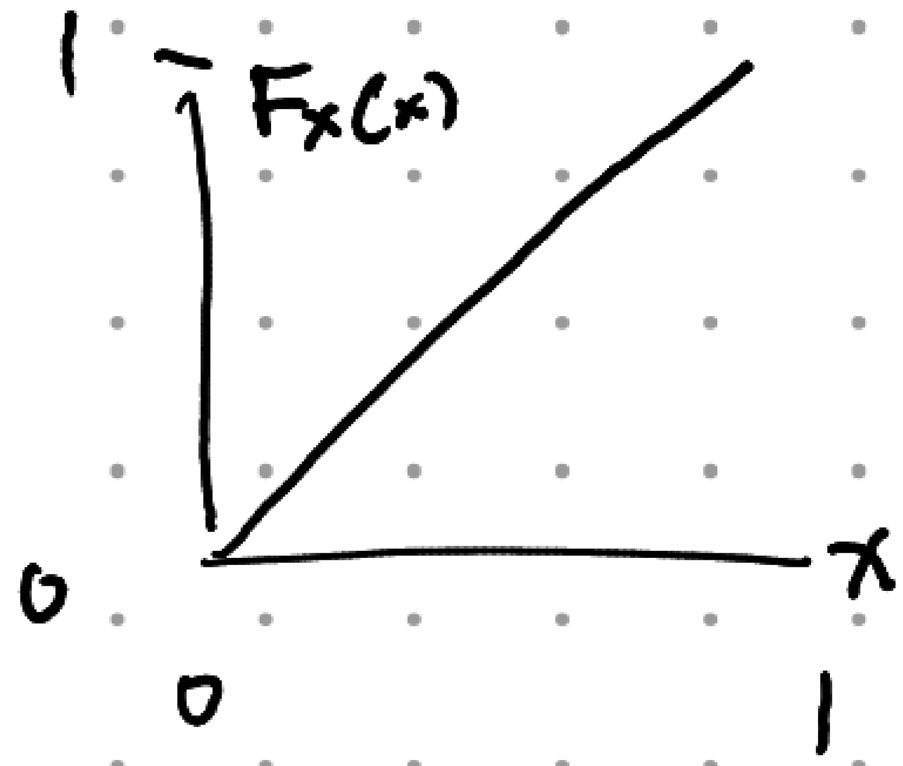
$F_X(x) = \mathbb{P}[X \leq x]$

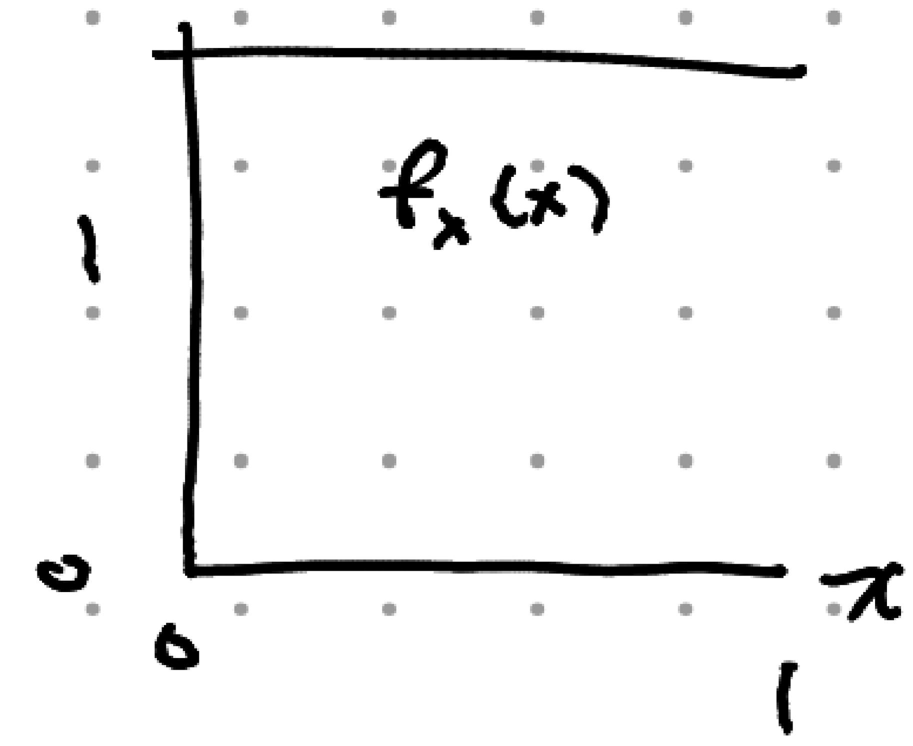
Ex. Write down CDF for coin example



Ex "equally likely from [0,1]"

$F_X(x) = x \qquad \mathbb{P}[a < X \le b] = \mathbb{P}[X \le b] - \mathbb{P}[X \le a]$





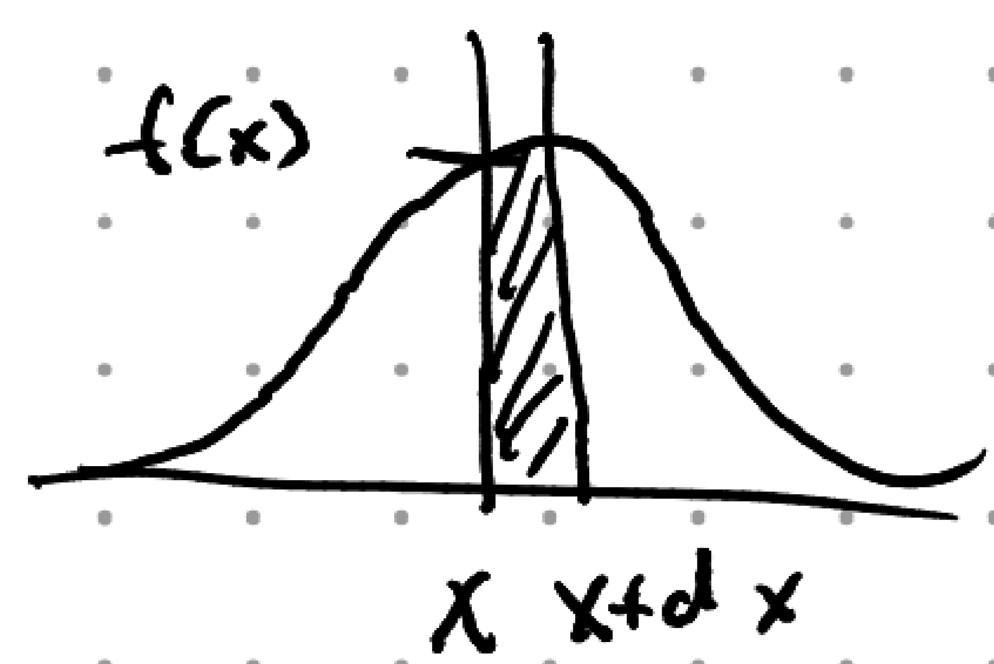
Density function

$f_X(x) = F_X'(x)$ (represents "relative" likelyhood of points)

From FTC: $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$

$\mathbb{P}[X \in A] = \int_A f_X(t)\,dt$

" $f_X(x)\,dx = \mathbb{P}[X \in [x, x+dx]]$ "



Aside.

For discrete RVs, $F_X'(x)$ not defined, but we can image we get the pmf.

Formally: delta distribution $\delta(x)$ defined by

$$\delta(x-t) = \frac{d}{dx}\begin{cases} 1 & x \ge t \\ 0 & x < t \end{cases}$$

$$\Longrightarrow \int_A \delta(x-t)\,g(t)\,dt = \begin{cases} g(x) & x \in A \\ 0 & \text{o.w.} \end{cases}$$

Theorem (informal)

$F : \mathbb{R} \to [0,1]$ s.t.

(i) F non-decreasing ($F(x_1) \leq F(x_2)$ if $x_1 \leq x_2$)

(ii) $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to \infty} F(x) = 1$

(iii) F right continuous:

$$\lim_{\substack{y \to x \\ y > x}} F(y) = F(x^+)$$

Then F is the CDF for some RV X

(also for some $\mathbb{P}$, since we could take $X(\omega) = \omega$)

Important RVs

- Unif(a,b)

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ 0 & \text{o.w} \end{cases}$$

- $N(\mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$

- Gamma, Beta, Chi-squared, Exponential