Homework 6 — Linear Algebra I

Instructions:

- Due April 23 at 11:59pm on Gradescope.
- You must follow the submission policy in the syllabus

Problem 1.

(a) Prove that $|\|x\| - \|y\|| \leq \|x - y\|$ for all $x, y \in V$.

(b) Suppose $S \in \mathcal{L}(V)$. Define $\langle \cdot, \cdot \rangle_1$ by $\langle x, y \rangle_1 = \langle Sx, Sy \rangle$ for all $x, y \in V$. Prove that $\langle \cdot, \cdot \rangle_1$ is an inner product if and only if S is injective.

Problem 2. In this problem we will consider the task of approximating a function with polynomials. This is at the core of approximation theory. One takeaway is will be that there are much better ways to do this than using a Taylor series.

Recall the Chebyshev polynomials are

$$
\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 16x^5 - 20x^3 + 5x \\
T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1 \\
T_7(x) &= 64x^7 - 112x^5 + 56x^3 - 7x \\
T_8(x) &= 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1 \\
T_9(x) &= 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x \\
T_{10}(x) &= 512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1
\end{aligned}
$$

and satisfy the orthogonality condition:

$$
\int_{-1}^{1} T_n(x)\, T_m(x)\, \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \begin{cases} 0 & \text{if } n \neq m, \\ \pi & \text{if } n = m = 0, \\ \frac{\pi}{2} & \text{if } n = m \neq 0. \end{cases}
$$

(a) Let $f(x) = \exp(x)$. For each $k = 0, 1, \ldots, 10$, find the degree k polynomial p_k which minimizes

$$
\int_{-1}^{1} (f(x) - p(x))^2 \frac{1}{\sqrt{1-x^2}} \mathrm{d}x
$$

and make a plot of $f(x) - p_k(x)$.

(b) Make similar plots with the error of the degree k Taylor series approximation.

(c) Make a plot of k (on the horizontal axis) versus $\max_{x\in[-1,1]} |f(x) - p_k(x)|$ (on the vertical axis). Put the vertical axis on a log-scale.

Add another curve for the error of the Taylor series approximation.

To compute the max, you can instead take a bunch of points (say 1000) equally spaced in $[-1, 1]$ and then take the max at those points.

(d) Repeat this for $f(x) = |x|$. But this time the Taylor series doesn't even exist, so you don't need to do that part.

(e) How do the rates of convergence (with respect to k) compare for the two functions? Why do you think this is?

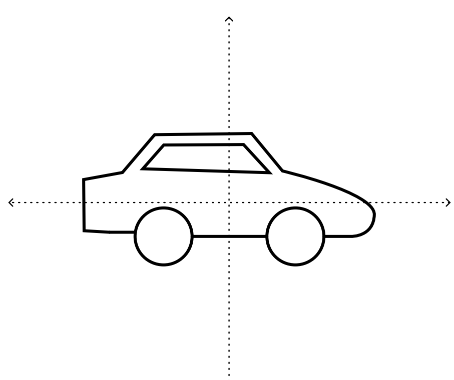
To compute integrals you could use Mathematica syntax `Integrate[ ChebyshevT[4,x] exp(x) , {x,-1,1}]` on Wolfram Alpha or some other tool.

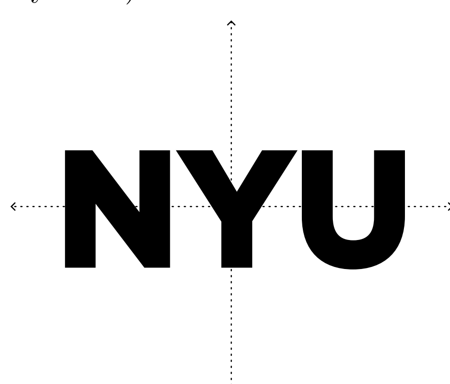
Problem 3. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\pi/4 & -\sin\pi/4 \\ \sin\pi/4 & \cos\pi/4 \end{bmatrix}$$

(a) Find a SVD of A. Hint: think about why the given factorization is not an SVD.

(b) Draw what A does to the following vector (here a vector is anything shown in black, and the dotted lines represent the x and y axes.):





Problem 4.

Recall $\|X\|_F^2 = \sum_{i,j} X_{i,j}^2$. The point of this problem is to relate the Frobenius norm of a matrix to its singular values.

(a) Suppose $\vec{X}$ is a $n \times m$ matrix. How does $\|\vec{X}\|_F$ relate to $\|\vec{X}^\mathsf{T}\|_F$?

(b) Suppose $\vec{X}$ is a $n \times m$ matrix. Write $\|\vec{X}\|_F$ in terms of the column-norms $\|[\vec{X}]_{:,i}\|_2$.

(c) Suppose $\vec{X}$ is a $n \times m$ matrix and $\vec{U}$ is a $n \times n$ orthogonal matrix ($\vec{U}^\top \vec{U} = \vec{I}$). Show that $\|\vec{U}\vec{X}\|_F = \|\vec{X}\|_F$. Hint: use (b) and show that $\|\vec{U}\vec{x}\|_2 = \|\vec{x}\|_2$ for any vector $\vec{x}$.

(d) Let $\vec{A}$ be a $n \times m$ matrix with SVD $\vec{A} = \vec{U}\vec{\Sigma}\vec{V}^\top$ and assume $n \geq m$. Prove that $\|\vec{A}\|_F = \sqrt{\sigma_1^2 + \cdots + \sigma_m^2}$, where σ_i are the singular values of $\vec{A}$

Problem 5.

(a) Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. (i) Prove that if $U \subseteq \operatorname{null} T$, then U is invariant under T. (ii) Prove that if $\operatorname{range} T \subseteq U$, then U is invariant under T.

(b) Define $T : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by $Tp = p'$. Find all eigenvalues and eigenvectors of T.

(c) Define $T \in \mathcal{L}\left(\mathcal{P}_4(\mathbf{R})\right)$ by $(Tp)(x) = xp'(x)$ for all $x \in \mathbf{R}$. Find all eigenvalues and eigenvectors of T.