Homework 5 — Linear Algebra I

Instructions:

- Due April 9 at 11:59pm on Gradescope.
- You must follow the submission policy in the syllabus

Problem 1 (Product and quotient space)**.**

(a) For a positive integer m, show that $V^m = \underbrace{V \times V \times \cdots \times V}_{m \text{times}}$ is isomorphic to $\mathcal{L}(\mathbb{F}^m, V)$. Do not assume V is finite-dimensional.

(b) Suppose $A_1 = v + U_1$ and $A_2 = w + U_2$ for some $v, w \in V$ and some subspaces U_1, U_2 of V. Prove that the intersection $A_1 \cap A_2$ is either a translate of some subspace of V or is the empty set.

(c) An equivalence relation is a binary relation that is reflexive, symmetric and transitive. Fix a subspace U of V. Show that $v \sim w$ if and only if $v - w \in U$ is an equivalence relation on V.

(d) Briefly explain how the previous problem relates to translates.

(e) Suppose U is a subspace of a finite dimensional vector space V. Prove that V is isomorphic to $U \times (V/U)$. (For a harder problem, you can replace the assumption V is finite dimensional with the assumption V/U is finite-dimensional)

Problem 2 (Duality)**.**

(a) Explain why each linear functional is surjective or is the zero map.

(b) Show that the dual map of the identity operator on V is the identity operator on V'.

(c) Suppose $m \geq 0$. What is the dual basis of $\{1, x-5, (x-5)^2, \ldots, (x-5)^m\}$ in $\mathcal{P}_m$?

(d) Suppose $T \in \mathcal{L}(V, W)$ and $w_1, \ldots, w_m$ is a basis of range T. Hence for each $v \in V$, there exist unique numbers $\varphi_1(v), \ldots, \varphi_m(v)$ such that

$$Tv = \varphi_1(v)w_1 + \cdots + \varphi_m(v)w_m,$$

thus defining functions $\varphi_1, \ldots, \varphi_m$ from V to $\mathbf{F}$. Show that each of the functions $\varphi_1, \ldots, \varphi_m$ is a linear functional on V.

Problem 3.

(a) Suppose $v_1, \dots, v_n$ and $v_1, \dots, u_n$ are such that $\operatorname{span}\{v_1, \dots, v_k\} = \operatorname{span}\{u_1, \dots, u_k\}$ for each k. Show that there exists an upper triangular matrix R such that

$$\begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix} R.$$

(b) Suppose we have a independent set of vectors $v_1, \dots, v_n$ and apply Gram-Schmidt to obtain an orthonormal set $u_1, \dots, u_n$ such that

> Set $u_1 = v_1/\|v_1\|$.
> For $k = 2, \dots, n$, set:
>
> $$\hat{u}_k = v_k - \langle v_k, u_1 \rangle u_1 - \cdots - \langle v_k, u_{k-1} \rangle u_{k-1}.$$
>
> and
>
> $$u_k = \hat{u}_k / \|\hat{u}_k\|.$$

(c) Show that the upper triangular matrix R you described in part (a) can be obtained from the coefficients computed by the Gram–Schmidt algorithm. That is, that you get the matrix R "for free" from the Gram–Schmidt algorithm.

Problem 4. Consider the vector space $\mathcal{P}_4$ of polynomials of degree at most 4. Define an inner product on $\mathcal{P}_4$ by

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x) \frac{1}{\sqrt{1-x^2}} \mathrm{d}x, \qquad \forall p, q \in \mathcal{P}_4.$$

(a) Verify this is an inner product.

(b) Apply the Gram-Schmidt process to the basis $\{1, x, x^2, x^3, x^4\}$ to obtain an orthonormal basis. You can use Wolfram alpha or similar to compute integrals, but should write down the integrals you are computing.

(c) Make a plot of the polynomials you computed and a different plot of the Chebyshev polynomials (up to degree 4). How do they compare?

Problem 5. (a) Suppose V is a real inner product space and $v_1, \ldots, v_m$ is a linearly independent list of vectors in V. Prove that there exist exactly 2^m orthonormal lists $e_1, \ldots, e_m$ of vectors in V such that

$$\operatorname{span}(v_1, \ldots, v_k) = \operatorname{span}(e_1, \ldots, e_k)$$

for all $k \in \{1, \ldots, m\}$.

(b) Suppose $C[-1, 1]$ is the vector space of continuous real-valued functions on the interval $[-1, 1]$ with inner product given by

$$\langle f, g \rangle = \int_{-1}^{1} fg$$

for all $f, g \in C[-1, 1]$. Let φ be the linear functional on $C[-1, 1]$ defined by $\varphi(f) = f(0)$. Show that there does not exist $g \in C[-1, 1]$ such that

$$\varphi(f) = \langle f, g \rangle$$

for every $f \in C[-1, 1]$.

(c) Suppose V is finite-dimensional. Suppose $\langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2$ are inner products on V with corresponding norms $\| \cdot \|_1$ and $\| \cdot \|_2$. Prove that there exists a positive number c such that $\|v\|_1 \leq c\|v\|_2$ for every $v \in V$.