Homework 4 — Linear Algebra I

Instructions:

- Due March 26 at 11:59pm on Gradescope.
- You must follow the submission policy in the syllabus

Problem 1. As mentioned in lecture, we can express abstract linear operatons on abstract linear spaces in terms of matrices and vectors. Here we will work through some of the math that would be required to manipulate polynomials on a computer.[1]

For conveneience, we will restrict to polynomials of degree at most 4.

We are used to working with the monomial basis:

$$m_0(x) = 1, \quad m_1(x) = x, \quad m_2(x) = x^2, \quad m_3(x) = x^3, \quad m_4(x) = x^4$$

For certain numerical reasons, we might also be interested in the Chebyshev Basis:

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x, \quad T_4(x) = 8x^4 - 8x^2 + 1$$

(a) Write down the differentiation matrices $D_m \in \mathcal{L}(\mathcal{P}_4, \mathcal{P}_4)$ the integration matrix (for a definite integral over $[a, b]$) $T_m \in \mathcal{L}(\mathcal{P}_4, \mathbb{R}^1)$ (with respect to the monomial basis for the input and output). The integration matrix should depend on a and b.

(b) Repeat (a) for the Chebyshev basis for the input and output spaces (call these matrices D_c and T_c).

(c) Write a the matrix M which converts a Chebyshev polynomial to a monomial polynomial, and the matrix N which converts a monomial polynomial to a Chebyshev polynomial.

Problem 2. Let D_m, D_c, M and N be the matrices defined in the previous problem.

(a) Describe in words what the following matrices are doing as operations:
   - $D_m M$
   - $D_c N$
   - MN
   - $T_m D_m D_m$

(b) verify by hand (meaning multiply out the matrices) the following:
   - $MN = I$
   - $NM = I$
   - $N D_m M = D_c$

[1] This is what makes stuff like numpy's `polynomial` module work.

- $MD_cN = D_m$
- $(D_m)^5 = 0$.

Problem 3. Implement the above operations in a computer program. I have provided started python code here: `https://colab.research.google.com/drive/1qqrpd-a7NnAo7uooQs1CVg`

Include the functions you implement in your submission. If you are using latex, you can use the `listings` package.

Problem 4.

(a) Suppose A is a $m \times n$ matrix and B is a $n \times p$ matrix. Prove $(AB)_{j,:} = (A)_{j,:}B$ for each $j = 1, \ldots, m$.

(b) Prove that matrix multiplication is associative: $(AB)C = A(BC)$ for any matrices A, B, C for which the products are defined.

(c) Suppose $v_1, \ldots, v_n$ is a basis for V and $w_1, \ldots, w_m$ is a basis for W. Show that if $S, T \in \mathcal{L}(V, W)$, then $\mathcal{M}(S + T) = \mathcal{M}(S) + \mathcal{M}(T)$.

(d) Prove that $(AC)^T = C^T A^T$ for any matrices A and C for which (AC) and $A^T C^T$ are defined.

(e) Suppose V is finite dimensional and $\dim(V) > 1$. Prove the set of noninvertible linear maps from V to itself is not a subspace of $\mathcal{L}(V, V)$.

(f) Suppose V is finite dimensional and $S, T \in \mathcal{L}(V, V)$. Prove that ST is invertible if and only if S and T are invertible.

(g) Suppose W is finite dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that $\text{range}(S) = \text{range}(T)$ if and only if there exists an invertible $E \in \mathcal{L}(V, V)$ such that $S = TE$.