Homework 2 — Linear Algbera I

Instructions:

- Due Feb 19 at 11:59pm on Gradescope.
- You must follow the submission policy in the syllabus

Problem 1 (Span)**.**

(a) Let $\vec{v}_1, \dots, \vec{v}_k \in V$. Prove $\text{span}(\vec{v}_1, \dots, \vec{v}_k)$ is a subspace of V.

(b) Show that $\text{span}(\vec{v}_1, \dots, \vec{v}_k)$ is the smallest subspace of V containing $\vec{v}_1, \dots, \vec{v}_k$.

(c) Find a list of four distinct vectors in $\mathbb{R}^3$ whose span equals

$$\{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}.$$

(d) Prove or give a counterexample: If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ is a linearly independent list of vectors in V and $\lambda \in \mathbf{F}$ with $\lambda \neq 0$, then $\lambda\vec{v}_1, \lambda\vec{v}_2, \dots, \lambda\vec{v}_m$ is linearly independent.

Problem 2 (Independence)**.**

(a) Let $\vec{v}_1, \dots, \vec{v}_k \in V$. Prove that if $\vec{v}_1, \dots, \vec{v}_k$ are linearly dependent if and only if there exists $i \in \{1, \dots, k\}$ such that $\vec{v}_i \in \text{span}(\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_k)$.

(b) Show that if the set of vectors $\vec{v}_1, \dots, \vec{v}_k$ are linearly independent, then none of the $\vec{v}_i$ are the zero vector.

(c) Prove or give a counterexample: If $\vec{v}_1, \dots, \vec{v}_m$ and $\vec{w}_1, \dots, \vec{w}_m$ are linearly independent lists of vectors in V, then the list $\vec{v}_1 + \vec{w}_1, \dots, \vec{v}_m + \vec{w}_m$ is linearly independent.

Problem 3 (Basis)**.**

(a) Show that a linear space spanned by a finite set of vectors has a basis. (This is Lemma 2 in Lax, but the proof is a bit too terse, so you should fill in the details. In particular, he writes "So we can drop that x_i" but does not explain what should happen after we drop x_i, and he writes that we can repeat this until they are linearly independent but does not justify why it will eventually become independent.)

(b) Show every vector space spanned by a finite set of vectors has a basis.

(c) Suppose $\vec{v}_1, \dots, \vec{v}_4$ is a basis of V. Prove that

$$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$

is also a basis of V.

(d) Prove or give a counterexample: If $\vec{v}_1, \dots, \vec{v}_4$ is a basis of V and U is a subspace of V such that $\vec{v}_1, \vec{v}_2 \in U$ and $\vec{v}_3 \notin U$ and $\vec{v}_4 \notin U$, then $\vec{v}_1, \vec{v}_2$ is a basis of U.

(e) Recall $\mathcal{P}_m(\mathbb{R})$ is the space of polynomials with real coefficients and degree at most m. Show that if $p_0, \ldots, p_m \in \mathcal{P}_m(\mathbb{R})$ are such that $\deg(p_i) = i$, then $p_0, \ldots, p_m$ is a basis for $\mathcal{P}_m(\mathbb{R})$.

Problem 4 (Dimesion)**.**

(a) Prove that if $X = Y_1 \oplus Y_2 \oplus \cdots \oplus Y_m$, then $\dim(X) = \dim(Y_1) + \dim(Y_2) + \cdots + \dim(Y_m)$.

(b) Recall $\mathcal{P}_4(\mathbb{R})$ is the space of polynomials with real coefficients and degree at most 4. Let $U = \{p \in \mathcal{P}_4(\mathbb{R}) : \int_{-1}^{1} p = 0\}$.
- Find a basis of U
- Extend the previous basis to a basis for $\mathcal{P}_4(\mathbb{R})$.
- Find a subspace W of $\mathcal{P}_4(\mathbb{R})$ such that $\mathcal{P}_4(\mathbb{R}) = U \oplus W$.

(c) Suppose U and V are both 3 dimensional subspaces of $\mathbb{R}^5$. Prove that $U \cap V \neq \{0\}$.